Swami Ramanand Teerth Marathwada University, Nanded, Maharashtra, India

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STUDY OF ITERATIVE METHODS IN FIXED POINT THEORY WITH APPLICATIONS

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❖ Introduction to Proposed Research Topic:

A fixed point of a function T : X→ X is a point x in X such that T(x) = x. Fixed point theory has a beautiful mixture of analysis, topology, fuzzy, game theory and geometry. Since from 1922 the theory of fixed points has been revealed as a very powerful and important technique for solving a variety of applied and technical problems in mathematical sciences and engineering. In particular fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, game theory, and physics, also in the method of successive approximations for proving existence and uniqueness of solutions of differential and integral equations. This method is associated with the names of such celebrated nineteenth century mathematicians as Cauchy, Liouville, Lipschitz, Peano, and specially Picard.

❖ Purpose of Carrying Out Proposed Research Work:

In the study of fixed point theory, the generalization of metric space is one of the interesting topics for many researchers. Some of the generalizations of metric space are 2-metric space, D-metric space, D^*-metric space, G-metric space, S-metric space, rectangular metric or metric-like space, partial metric space, fuzzy-metric space, probabilistic metric space, cone metric space, modular metric space, etc.

So we attempt to generalise a new type of iterative methods by using different iterative methods in above metric space with new metric space along with topology of this space. We prove existence of some fixed point theorems and its applications.

❖ Survey of Literature and Development in the relevant Subject:

Metric spaces form a natural environment for exploring fixed points of single and multi-valued mappings. Metric fixed point theory has been an area of vigorous scientific activity since the basic result of Banach in 1922. Applications of such results cover several areas of mathematics and other sciences. It may be noted that the use of triangle inequality in metric arguments is of extreme importance since it implies, among other things, the following:

(1) The metric is continuous in both variables.

(2) The respective topology is Hausdorff.

(3) In particular, a sequence may converge to at most one point.

(4) Each open ball is an open set.

(5) Each convergent sequence is a Cauchy sequence.

Due to the nature of mathematics science, there have been many attempts to generalize the metric setting by modifying some of the axioms of metric spaces. Thus, several other types of spaces have been introduced and a lot of metric results have been extended to new settings.

One of the interesting generalizations of the notion of metric space was introduced by Branciari in 2000, where the triangle inequality was replaced by a so-called rectangular (or quadrilateral) inequality, involving four (or more) instead of three points. It was not immediately observed that such spaces called rectangular or generalized metric spaces

In 1930, Brower's fixed point theorem was generalized to Banach spaces. This generalization is known as Schauder's fixed point theorem and this result further generalized by so many researchers. Also in 2007 Huang and Zhang introduced spaces named as cone metric spaces and also introduces the definition of convergent and Cauchy sequences in terms of interior points of the underlying cone. They also proved some fixed point theorems in cone metric spaces [18, 19].

In recent years, there has been increasing interest in the study of fixed points and common fixed points of mappings satisfying contractive conditions of integral type.

Towards the end of 19^{th} century it becomes clear many mathematicians that Riemann integral should be replace by some other types of integral, more general and more flexible better suited for dealing the limit process. Since that time different types of nonlinear integrals are successfully discussed by various mathematicians and developed some theories about the integral, differential equations. In this research work we shall studying the existence, uniqueness of the solution of the nonlinear measure, Integro-Differential equation. The topic of nonlinear measure Integro-Differential equations has been categories into branches and sub-branches. The theory of nonlinear differential equations has attracted several mathematicians and at present study lies in the fact that many of the physical and biological or social phenomena are nonlinear in nature and can be modelled on the nonlinear ordinary and partial differential, integral and Integro-Differential equations.

❖ Importance of the Study:

The study of metric fixed point theory plays an important role because the study finds applications in many important areas as diverse as differential equations, integral equations, operation research, mathematical economics, and engineering. Also it is very useful in nonlinear mathematical analysis with applications; because of this high importance a huge amount of research is carrying out last 90 years and in future also.

a) International Status:

One of the most dynamic area of research of the last 50 years, fixed point theory plays a fundamental role in several theoretical and applied areas, such as nonlinear analysis, integral and differential equations and inclusions, dynamic systems theory, mathematics of fractals, mathematical economics (game theory, equilibrium problems, optimization problems) and mathematical modelling. This thematic series will present relevant works related to the theory of fixed points and its various applications to pure, applied and computational mathematics. Special attention will be paid to the most important theories developed by following researchers:

Prof. Ioana Rus and Prof. Vasile Berinde [20], [21] the Cluj-Napoca Fixed Point Theory. The Picard operator theory, the fixed point structure theory and other aspects of fixed point theory.

Prof. R. P. Agarwal [11] worked in Florida Institute of Technology, USA along with Dr. O'Regan on Fixed Point Theory for Lipschitzian-type mappings with applications to differential and difference equations.

Prof. William Arthur Kirk is one of the founders of the modern theory of metric fixed points. With more than 175 works in the field of fixed point theory influenced the development of this flourishing field in a decisive way. The very first and, maybe, most notable result in fixed point theory was given by his famous result on the existence of fixed points for non-expansive mappings in reflexive Banach spaces with normal structure, published in Amer math monthly in 1965. This wonderful result opened a large gate to a whole new world to be explored, where geometry of the Banach spaces, non-expansive mappings and fixed points were interconnected in a very unexpected and fruitful way. Later, in some papers with Kazimierz Goebel, weaker forms of non-expansivity (such as asymptotically non-expansive mappings) were also introduced

and studied, aspects which are even today a source of inspiration for new generations of mathematicians.

Prof. Ulrich Kohlenbach (Technische Universitat Darmstadt, Germany) worked on the computational content of the Krasnoselskii and Ishikawa fixed point theorems. He also gives the contribution in bounds on iterations of asymptotically quasi-non-expansive mappings.

Prof. Poom Kumam (Fixed Point Laboratory, Thailand) worked on Fixed Point theorem and random fixed point theorems for set-valued non-self-mappings, and also proved some fixed point Theorems for non-expansive mappings in Modular Spaces.

b) National Status:

The field of Fixed Point Theory is enriched due to the great contribution of many researchers throughout India, some of them are:

Prof. R. Kannan proves the existence of new type of Fixed Point Theorems.

Prof. Tanmoy Som (IIT Banaras Hindu University, Varanasi) worked on some generalizations of fixed point theorems.

Prof. Dr. Javid Ali (Aligarh Muslim University, Aligarh) worked on Fixed Point Theory and its Applications with Functional Differential Equations.

c) Regional Status:

A lot of contribution is also done by researchers at our state level universities.

In which major work is going on Dr. B. A. Marathwada University, Aurangabad, Swami Ramanand Teerth Marathwada University, Nanded on particularly random and hybrid fixed point theorems and its applications to integral and differential equations with abstract measure spaces.

d) Scope of Study:

Fixed Point theorems in topology and nonlinear functional analysis are usually based on certain properties such as completeness, continuity, contractiveness, etc. that the operator, considered as a single entity must satisfy. By combine some properties with new operators of metric spaces and generalizing fixed point theorems in it we can obtain some related results which may be further useful for finding applications. Many

nonlinear equations can be solved using Fixed Point Theorems; in fact Fixed Point Theorems have applications in nonlinear Integral equations, Differential equations, Difference equations, Optimization theory, Game theory, Fuzzy theory and so on. In a wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable map. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences. Fixed point results provide conditions under which maps have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology, fuzzy and geometry. Over the last 90 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics.

❖ Objectives of the proposed research work:

- To Study in details some results in Fixed point theorems as Banach contraction principle, Brouwer's fixed point theorem, Schauder's fixed point theorem, Kannan's fixed point theorem, etc.
- To Study in details metric spaces and generalized metric spaces and generalize various fixed point theorems in it.
- Now defining a new contraction mapping and prove some generalization of fixed point theorems in it with supporting examples, before that mentioning a weak contraction (α, L) map introduced by Berinde [20, 21] and proved that various contractive maps converted in the form of (α, L) weak contraction and almost all fixed point results are proved, here now first mention a Berinde weak contraction type common fixed point theorem in function weighted metric space and then generalize that result in new weak contraction map.
- One-step Mann iteration process to construct superior Julia and Mandelbrot sets for n^{th} degree complex polynomials $P_r(x) = x^n + r$, first used by Rani and Kumar [22, 25] in which the Mann orbit, for any $x_0 \in \mathbb{C}$, is a sequence $\{x_i\}$ defined as $x_{i+1} = (1 - \alpha)x_i + \alpha P(x_i)$, for $i \geq 0$ and $\alpha \in (0, 1]$.
- A two-step Ishikawa iteration process used by Rani and Chauhan [22] and Chauhan et. al. [23] in the year 2010 for the study of relative superior Julia and relative superior Mandelbrot sets, respectively. The dynamics of the n^{th} order complex

polynomial for non-integer values were searched [24] also they construct new Julia and Mandelbrot sets using Ishikawa orbit, for any $x_0 \in \mathbb{C}$ is a sequence $\{x_i\}$ defined as $\begin{cases} x_{i+1} = (1-\alpha)x_i + \alpha P y_i, \\ y_i = (1-\beta)x_i + \beta P x_i, \end{cases}$ where, $i = 0, \ldots, \infty$ and $\alpha, \beta \in (0, 1]$.

- The three-step Noor iteration process for Julia and Mandelbrot sets search out by Rani and Ashish [25], in the Noor orbit for any $x_0 \in \mathbb{C}$ is a sequence $\{x_i\}$ defined as

$$\begin{cases} x_{i+1} = (1-\alpha)Px_i + \alpha_i P y_i, \\ y_i = (1-\beta)Px_i + \alpha P u_i, \\ u_i = (1-\gamma)Px_i + \gamma P x_i, \end{cases}$$ where, $i = 0, \ldots, \infty$ and $\alpha, \beta, \gamma \in (0, 1]$.

- Applying obtained fixed point theorems result in generalized metric spaces to obtain existence and uniqueness of solutions of nonlinear measure Integro-Differential equations.

❖ Hypothesis:

In the year 1989, Bakhtin [17] introduced the concept of b-metric space in which the triangle inequality is made by a more general form by introducing a scale factor which is a real number greater than or equal to one. By doing so the ordinary metric space becomes a special case of b-metric space when scale factor is equal to one.

Introduction of b-metric space opens a gate to further generalization of metric space. Due to the introduction of b-metric space, many generalizations of metric space came into existence. Many researchers obtained fixed point theorems for various mapping in different metric spaces. Among them are G-metric spaces, S-metric space with b-metric space which are a generalization of metric spaces. Mention can be made about Gb-metric space, Sb-metric space, similarly rest are b2-metric space, cone b-metric space, partial b-metric space, fuzzy b-metric space, rectangular b-metric space etc [11].

We define the concept of soft new metric, according to soft element, and describe some of its properties. Also, we give a relation between the soft metric and soft new metric. Further, we introduce soft new ball and get its basic features, we investigate soft convergence and soft continuity with the help of soft new metric. Then, we prove existence and uniqueness of fixed point in soft new metric spaces. Finally we try to apply these results in integral and differential or difference equations [12].

❖ Limitation of the work:

In this proposed research topic as there are various iterative methods related to fixed point results and metric spaces developed by researchers so we take some of them standard and common results and trying to obtain and proving generalized result of fixed point theory with uniqueness, existence and properties. Also these study results we apply to some current emerging applied special sub-branches of Differential, Integral, Difference, Integro-Differential equations.

❖ Research Methodology:

Following is the research methodology to be used

- Collection of references through research journals/internet.
- Understanding the knowledge about current research topic by participating in the topic related conferences, workshops, seminars.
- By studying references finding new results.
- Finding out of new results with supportive examples.

❖ Expected Outcome:

We hope the work of this research will be useful to the engineers, scientists and mathematicians, researchers for the international, national level. That research work pursuit their studies in solving some scientific problems in dealing games or discontinuity. The present work will also be useful to the research in the field of the differential equations, integral equations, and Integro-Differential equations. It has a nice scope for further development in the future. Also this study will help researchers to upgrade and support the further studies on soft topology, soft metric and fixed point theory to carry out a general framework for their applications in real life. In future researchers and scientist will be used that theory in the different field of applications.

❖ Proposed work plan/Year-wise plan of work and targets to be achieved:

➢ I Year plan:

The collection of basic materials required for this research. This includes collection of Journals, Books and some research papers related to the research. We would like to visit few libraries and institutions like IITS, IISC, IISER/NISER, TIFR and some universities for collection of literature. In this year, we will complete course work. Two progress reports will be submitted at the end of this year.

➢ II Year plan:

The collected material will be proceeded and attempt will made to discover the finding in the direction of research different methodology, tools, techniques and guidance will be apply for new research problems. Some national and international visits for the guidance of research work verification, methodology verification. To the study of new research problem in different form and it will be submitted to research journals for publication. Two progress reports will be submitted at the end of this year.

➢ III Year plan:

The outcome of our research work will be presented in national and international gatherings such as, conferences, seminars, and workshops during this year. Also we discuss the present new research work findings with various researchers, lecturers, students. The research work will be published in the several reputed journals. The printing and binding of project work. The papers of project work based on existence theory will be communicated to the different national and international journals for the publication. Two progress reports will be submitted at the end of this year and submission of final synopsis will be done in this year to the authority.

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